The Interplay of Text, Symbols, and Graphics in Mathematics Education

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Abstract:

Mathematics textbooks and other teaching materials are almost always multi-modal, containing text, symbolic notation, and graphics. The language of mathematics is precise and technical, the diagrams and graphs make extensive use of implicit conventions, and mathematical notation is information dense, often non-linear, and may occupy a cognitive space somewhere between text and graphics. Thus, to effectively design mathematics teaching materials, it is necessary to understand how learners interact with the multi-modal nature of mathematics.

This paper begins by surveying the literature with respect to the verbal component of mathematics text and follows that with discussions of the visual and symbolic components. Finally, research that addresses the interactions amongst these modes is considered, and some areas requiring further research are identified.

Key Words:

Mathematics education, mathematics text, equation reading, language and mathematics, visual images.

Introduction

Mathematics textbooks and other teaching materials are almost always multi-modal, containing text, symbolic notation, and graphics. The language of mathematics is precise and technical, the diagrams and graphs make extensive use of implicit conventions, and mathematical notation is information dense, more often than not, non-linear, and may occupy a cognitive space somewhere between text and graphics. The reform movement in mathematics education endorses the use of multiple representations for the introduction of new concepts and objects so that students are expected to integrate information obtained graphically, analytically, and numerically to form robust schema which they can apply to novel situations. Diagrams must be understood and related to the text that accompanies them. Symbolic expressions need to be created to model real world problems and then graphs constructed from the symbolic models. The graphs may then guide an analytic solution that needs to be interpreted in terms of the original practical problem. This highly complex interplay between the modes is seldom taught explicitly, but is learned (or often not learned) through a kind of apprenticeship.

Thus, to effectively design mathematics teaching materials, it is necessary to understand how learners interact with the multi-modal nature of mathematics. How do they read symbolic expressions, and what meaning do they attach to them? How do they relate textual information to the symbols and the symbols to the graphs? To address these kinds of questions, we need to understand the multi-modal nature of mathematics on many levels, taking into account the internal structures of each mode and the interplay amongst them.

This review begins by surveying the literature with respect to the verbal component of mathematics text and follows that with discussions of the visual and symbolic components. Finally, research that addresses the interactions amongst these modes is considered, and some areas requiring further research are identified.

Verbal component

The language of mathematics is informationally dense and structurally complex, leading several authors (Meaney, 2005, Morgan, 2006, and O’Halloran, 2005) to use Michael Halliday’s systemic functional linguistics (SFL) to investigate its structure. The most prominent characteristic, identified by all three researchers, is the use of dense noun phrases (nominal groups) to express complex ideas. Actions are rephrased as nominal groups, and the resulting text contains verbs which describe relationships rather than actions. For example, the statement “a third degree polynomial with real coefficients has at most three real roots” links two nominal groups with the relational verb “has.” Additionally, specialized terminology is often in conflict with common usage, and extensive use of logical connectives results in the complex sentences characteristic of mathematical text (Meaney, 2005). While these characteristics make the acquisition of the mathematical register problematic for many students, they support a precise and concise description of mathematical ideas, permitting the language to function effectively for the development of mathematics (O’Halloran, 2005). Halliday’s SFL also provides these researchers with a sociological perspective on the linguistic structure of mathematics text. Meaney points out that the creation of nominal groups hides human action while strengthening the impression of mathematics as objective truth. Additionally, mathematical ideas are treated as objects, leading to an impression that mathematics is an objective reality to be discovered. O’Halloran agrees with this assessment and writes of the language removing the human dimension. The students are placed in a passive relationship with mathematics, seeing it as something they cannot penetrate or create.

Herbel-Eisenmann and Wager (2007) use critical discourse analysis to examine the ways mathematics textbooks influence students’ experiences of mathematics. They report that first person pronoun use is almost entirely restricted to *we*, and the reference appears to be to an omniscient mathematics community, rather than to a partnership of writer and reader. The pronoun *you* is used to tell the readers about themselves – you think, you find – leaving the author in full control of the common knowledge. This pronoun use, they assert, leads to a sense of depersonalization and disconnection.

Additionally, the reader is placed in a passive stance by language in which “the graph shows” rather than “the reader sees.” Morgan (2006) has found that students themselves often adopt the same language. A lack of hedging—the words might, may, and could are uncommon in mathematics text—and the frequent use of words implying certainty (must, will, clearly) contribute, according to Herbel-Eisenmann and Wager, to an absolutist view of mathematics itself. Moreover, the status of real life is reduced to examples that simply provide opportunities for the use of “real” mathematics, leading, they argue, to decontextualization. Meaney (2005), who also observed this demotion of real life, asserts that it reinforces the view that the context of mathematics is the mechanics of mathematics.

The overall effect is dense, complex sentences which obscure the presence of people, distance the reader from the author, and portray the student as passive and mathematics as impersonal.

Visual Component

The primacy of the visual in mathematics has been observed by several researchers. Representing abstract phenomena that cannot be directly seen, the graphics appearing in mathematical text are, according to Cook (2006), essential. Sfard (2008) agrees, arguing that the acquisition of mathematical knowledge is visually mediated through diagrams, graphs, and drawings, which she classifies together with algebraic notation as symbolic artefacts. Presmeg (2006) also groups the symbolic and graphical together as mathematical signs, although in different registers, and agrees with Sfard that the primary modality in mathematics is visual.

The acquisition of mathematical meaning from visual displays is complex and difficult. O’Halloran emphasises that learners really cannot see what is intended unless they already understand the conventions being used and know what it is they are looking for. She contends that, as a result, the visual cannot be treated autonomously, but must be considered in context. Cook (2006) comes to a similar conclusion, suggesting that guidance is important in learning from visuals, as learners must use prior knowledge to select what is relevant before they can use that information to develop mental models. Watkins, Miller, and Brubaker have shown that visual images do not necessarily promote meaningful learning because students’ inferences are often far from what is intended. In one experiment over 60% of the students constructed their own elaborate, and erroneous, explanations of illustrations rather than reporting simple observations. Watkins et al. found that, without guidance, students do not make a connection between text and visuals and so base their interpretations on the surface characteristics of the graphics. A similar result was reported by Northcut (2007) who discovered that when learners lack a vocabulary for describing images they turn to evaluation and commentary on aesthetics rather than content. Presmeg (2006) suggests that visuals alone are generally too specific, presenting a single concrete picture which may obstruct necessary generalization. For example, a common misunderstanding of functions as always being continuous likely arises from the almost universal use of smooth, continuous graphs as generic illustrations. On the other hand, O’Halloran observes that visuals are more intuitive than are symbolic expressions, allowing their use as a means to experiment and synthesize.

Herbel-Eisenmann and Wager (2007) found that visuals contribute as much as text to a disembodied, decontextualized, and detemporalized mathematics. Few humans are included, and those that do appear are generic or disembodied—a stick figure or a hand pointing. Images of people doing mathematics seldom appear. People are, instead, the subject of mathematics; their heights are measured, their earnings calculated, their velocities observed. The result is to position the reader in a passive relationship with the subject. Northcut (2007) also takes a sociological approach, calling for the application of Feenberg's critical theory of technology to scientific and technical illustrations in educational material. This theory provides three approaches to technology: instrumental, in which technology is seen as a mere tool; substantive, which considers the effect of technology on the environment; and critical, which addresses the social control of technology. Northcut contends that analytic tools are needed to get at the complexity of images and their interpretations and that the critical theory could provide understanding of the way the power of images is used. Unfortunately she does not go beyond advocating the development of this theoretical approach into any substantive demonstration of how Feenberg's theory might work in this application. O'Halloran's (2005) choice to adapt of O'Toole's systemic functional framework to analyse the choices made in visuals appears more promising as it allows her to assess the social and educational implications of mathematical graphics, demonstrating the absence of human and social content from most illustrations and highlighting the socially problematic content of others. Cook (2006) expresses regret that educational research has not yet developed a theory of how to design teaching materials in mathematics that incorporate visuals and asserts any such theory would need to rely on understanding the structure of the various kinds of visuals, cognitive psychology, and sociological analysis. Unfortunately, she provides no direction toward such a theory.

Taken together, the research clearly indicates the importance and complexity of visual mathematics, identifying both cognitive and sociological issues that need to be addressed.

Symbolic component

The symbolic expressions of mathematics can be read like language and have their own syntax, but are also two dimensional and so share some of the characteristics of diagrams. Simply reading a symbolic expression is a complex task. Gillan, Barraza, Karshmer, and Pazuchanics (2004) report that, as with written English, expressions are generally read from left to right, but with a significant amount of backtracking. Novices pay little attention to the parentheses used to structure expressions, choosing instead to focus on processing individual operations. Other researchers have found that students respond strongly to the visual structure of symbolic mathematics, independent of the semantic content. As reported by Noguiera de Lima and Tall, (2008) learners even mentally pick up the symbols and move them around based on rules of motion that they have memorized. For example, in simplifying expressions such $3a+2b+2a$, students describe "picking up" the $2a$ and moving it next to the $3a$ before combining them to get $5a$ (p.7). Kirshner and Awtry (2004) found that rules are often inferred based on visual structure, and visually salient rules are over generalized, so that $(a+b)c = ac + bc$ may lead learners to an erroneous assumption that $\sqrt[c]{a+b} = \sqrt[c]{a} + \sqrt[c]{b}$ (p. 5). As well, they found that a sense of animation develops, with the left side of an equation transforming into the right, in a way that adds a temporal dimension. Similar results, especially with respect to the sense of animation, were reported by Landy and Goldstone (2007).

Sfard (2008) and Presmeg (2006) both argue that mathematical symbolism is more than just a language with which to record mathematics. Rather, since the objects of mathematics are not accessible to the senses except through their symbols, the symbols act as objects in mathematics, and we work with these signs as if they were the objects signified. Mathematics, Sfard says, is an autopoietic system, creating the very things that it talks about. At the core of her discussion is a description of the process by which mathematical procedures are reified to become the objects represented by the symbols, a process of that is mediated symbolically.

O'Halloran (2005), on the other hand, describes mathematical symbolism as an information dense language, although with strategies for organizing meaning that differ from those of natural language. While the symbols remove the human dimension, they increase the operational, relational, and existential meaning, and they can be operated on to solve problems without recourse to the world. Using Halliday's SFL, she identifies information dense nominal groups and relational, rather than active, verbs in the symbolic expressions of mathematics. For example, the statement

$$\frac{d}{dx}(2x^2 - 3x) = 4x - 3$$

consists of two nominal groups linked by the relational verb "equals." The actions of multiplying, squaring, subtracting, and calculating the derivative are hidden within the nominal groups. Meaney (2005), also using SFL, identifies the same structures and asserts that the high lexical density of symbolic mathematics allows great flexibility in the way symbolic expressions can be used. However, this complexity means readers need specific skills to unpack the meaning. Both Meaney and O'Halloran note that mathematical symbolism is not taught linguistically and assert that this adds to the difficulty students have in mastering it.

The symbolic component of mathematics text is not well understood. General agreement as to whether symbolic mathematics is primarily linguistic, visual, or a hybrid of these modes has yet to emerge, and integration of results from the linguistic and cognitive perspectives is lacking. As well, little research has yet been undertaken into the two dimensional visual structure of mathematical expressions.

Multimodal analysis

Mathematics text is essentially multimodal, and researchers are in substantial agreement that the three modes, verbal, visual, and symbolic, are all necessary for the learning of mathematics. O'Halloran (2005) asserts that the different modes provide different ways for making meaning in mathematics and that the division of work is complementary. Schleppegrell (2007) agrees that the modes must work together to construct meaningful mathematics, language providing the context for problems, symbolic mathematics describing patterns or relationships, and drawings providing a connection to the physical world. What sets mathematical discourse apart from other discourses, according to Sfard (2008), is that no one mode would suffice to convey meaning. Presmeg (2006) goes further in suggesting that conversion amongst modes is useful to combat compartmentalization and facilitate the conversion of processes into objects. For example, before students can make sense of operations, like differentiation, on polynomials they must see polynomials such as $4x^2 + 3x + 2$ as objects rather than as instructions for carrying out calculations. This reification is, Presmeg asserts, a necessary step in the creation of the nominal groups that make up much of mathematic language and symbolic representation. However, she notes that the processes involved in conversion are not well understood. To really get at how learners are able to make meaning of what they read and see, we need to understand the relations amongst the modes.

Researchers have reported that the use of multiple modes in mathematics texts can raise problems as well as facilitate learning. Berends and van Lieshout (2009) have shown that the strength of the linkage between information in illustrations and accompanying text affects learner's ability to solve problems. In a study of students working on calculus problems, Haciomeroglu and Aspinwall (2007) confirmed that without support from analytic thinking, images can be a hindrance rather than a help. However, according to Cook (2006) and to Mayer and Moreno (2003), novices have difficulty coordinating representations. Mayer and his colleagues and students have researched text and image relations for many years, providing a large body of work informed by cognitive psychology. Visuals and text use separate perceptual and cognitive pathways, they conclude, so that visuals need to be explicitly linked with the verbal component. Without this linkage, negative effects (lowered retention and transfer) can occur due to the increased cognitive load arising from split attention and double processing of information. Unfortunately, this body of work does not address the inclusion of symbolic content.

Some form of theoretical framework is needed if we are to make real progress in understanding the multimodal nature of mathematics learning. Sfard's (2008) discourse analysis provides insight into the structure of mathematics and the process of nominalization but does not include any framework for understanding the cognitive interactions amongst modes. O'Halloran (2005) provides useful tools for analysing mathematical text and symbolic expressions, but her treatment of the visual component is from a different point of view and her discussion of integration across modes provides a set of observations that seem too loosely associated to be useful for our purposes.

Matinec and Salway (2007) provide an interesting framework for analysing image-text relations, drawing on Barthe and Halliday to create a system based on perceivable characteristics and which is applicable to many genres. Two major subsystems are identified: status relation in terms of equality, subordination, and independence; and logico-semantic relation in terms of expansion, extension, and enhancement. This classification system could prove useful in illuminating some of the conflicting results in the experimental work of Mayer and his colleagues and of Berends and van Lieshout and has the potential to address sociological issues as well.

Further research

Further research is needed into the cognitive psychology involved in reading multimodal mathematics text. A vigorous and well developed research program exploring the relations between visuals and text already exists and is likely to continue. However research in the area of the cognitive processing of symbolic expressions is not extensive. Before we can really get at the interaction of symbolic mathematics with text and visuals, we need to know a great deal more about how symbolic expressions are processed, how people actually read equations, and whether equations are fundamentally linguistic, visual, or occupy a space between these two.

A second research direction is toward the construction of a single overarching framework that could be used to analyse the interactions amongst all three modes. Martinec and Salway's framework for text-image relations could prove a good candidate to expand to include symbolic mathematics. Such a framework would be helpful for researchers investigating either the cognitive or the sociologic aspects of mathematics text.

Conclusion

The abstract nature of mathematics combines with the multimodal nature of its texts to make any understanding of educational material in this subject extremely complex, requiring tools from sociology, linguistics, and cognitive psychology, at the very least. Much progress has been made, especially by Mayer and his colleagues working from the cognitive perspective and by O`Halloran and Sfard, who address the issues from backgrounds in linguistics and discourse analysis. A fuller understanding depends upon continued research and analysis by researchers from diverse backgrounds.

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